

# Perturbing free motions on hyper-spheres without degeneracy lift

A. Pallares-Rivera<sup>1</sup>, F. de J. Rosales-Aldape<sup>2</sup>, M. Kirchbach<sup>3</sup>

Institute of Physics  
Autonomous University of San Luis Potosí  
Av. Manuel Nava 6, University Campus  
SLP 78290 San Luis Potosí, México

**Abstract:** We consider quantum motion on  $S^3$  perturbed by the trigonometric Scarf potential (Sarf I) with one internal quantized dimensionless parameter,  $\ell$ , the ordinary orbital angular momentum value, and another, continuous parameter,  $b$ , through which an external scale is introduced. We argue that a loss of the geometric hyper-spherical  $so(4)$  symmetry of the free motion occurs that leaves intact the unperturbed hydrogen-like degeneracy patterns characterizing the spectrum under discussion. The argument is based on the observation that the expansions of the Scarf I wave functions for fixed  $\ell$ -values in the basis of properly identified  $so(4)$  representation functions are power series in the perturbation parameter,  $b$ , in which carrier spaces of dimensionality  $(K + 1)^2$  with  $K$  varying as  $K \in [\ell, N - 1]$ , and  $N$  being the principal quantum number of the Scarf I potential problem, contribute up to the order  $\mathcal{O}(b^{N-1-K})$ . Nonetheless, the degeneracy patterns can still be interpreted as a consequence of an effective  $so(4)$  symmetry, i.e. a symmetry realized at the level of the dynamic of the system, in so far as from the perspective of the eigenvalue problem, the Scarf I results are equivalently obtained from a Hamiltonian with matrix elements of polynomials in a properly identified  $so(4)$  Casimir operator. The scheme applies to any dimension  $d$ .

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## 1 Symmetry and degeneracy: Introductory remarks

Symmetry and degeneracy are two concepts which one traditionally associates with the basics of the quantum mechanics teachings. One may think of the degeneracy with respect to the magnetic quantum number,  $m$ , of a quantum level of a given angular momentum  $\ell$  which is  $(2\ell + 1)$ -fold, and typical for the states bound within all central potentials. A more advanced example would be the degeneracy of the states within the levels describing the quantum motion within the Coulomb potential of an electron without spin, which is  $N^2$ -fold with  $N$  standing for the principal quantum number of the Coulomb potential problem. In the first case, the degeneracy is due to the rotational invariance of the three-dimensional position space, which requires conservation of angular momentum, and demands the total wave functions of the central potentials to be simultaneously eigenfunctions of  $\mathbf{L}^2$ , and  $L_z$ , with  $\mathbf{L}$  standing for the angular momentum pseudo-vector, and  $L_z$  for its  $z$ -component. As long as  $\mathbf{L}^2$  acts as the Casimir invariant of the  $so(3)$  algebra, the degeneracy with respect to the magnetic quantum number (the  $L_z$  eigenvalues  $m \in [-\ell, +\ell]$ ) is attributed to the rotational invariance of the Hamiltonian. The second case is bit more involved in so far as in order to explain the larger  $N^2$ -fold degeneracy, one needs to invoke the higher  $so(4)$  symmetry algebra underlying the Coulomb potential problem by accounting for the constancy of the Runge-Lenz vector next to that of angular momentum [1].

<sup>1</sup>e-mail: pallares@ifisica.uaslp.mx

<sup>2</sup>e-mail: r\_felipedejesus@yahoo.com.mx

<sup>3</sup>e-mail: mariana@ifisica.uaslp.mx

The list can be continued by some more examples, a popular one being the case of the Pöschl-Teller potential,  $V_{PT} = -\frac{\lambda(\lambda+1)}{\cosh^2 \eta}$ . In [2] it has been noticed that the free quantum motion on the one-sheeted two-dimensional hyperboloid,  $x^2 + y^2 - z^2 = 1^2$ , an  $AdS_2$  space [3], transforms, upon an appropriate change of variables, into the one-dimensional (1D) Schrödinger equation with same potential. As long as the kinetic-energy Laplace-Beltrami operator on the hyperboloid is proportional to the Casimir operator of the  $so(2, 1)$  isometry algebra of the  $AdS_2$  surface, the Hamiltonian of the free motion on the curved space is equivalent (up to additive constant) to the aforementioned Casimir invariant and the spectrum of the potential under discussion correspondingly exhibits spectral patterns characteristic for  $so(2, 1)$ . In consequence, also the spectrum of the related 1D-Schrödinger equation with the Pöschl-Teller potential will be classified according to the irreducible representations of the same algebra and will carry patterns identical to those of the free motion on  $AdS_2$ .

The idea, that the spectrum of a Schrödinger equation with a given potential exhibits certain Lie algebraic degeneracies because in some appropriately chosen variables it becomes identical to the eigenvalue problem of the Casimir operator of the isometry algebra of a curved surface, the former not necessarily being unitarily equivalent to its canonical representation, has been further elaborated and generalized by many authors (see [4] for a review). It has become known in the literature under the name of “symmetry algebra of a potential”, or, simply, “potential algebra”. The potential algebra concept attributes to an underlying algebraic symmetry the degeneracy patterns of a potential. Predominantly, the  $su(1, 1)$  symmetry of the Natanzon-class potentials has been extensively studied within this context, for example in refs. [5], [6], [7], [8], among others, but also the  $so(4)$  symmetry of the trigonometric Pöschl-Teller potential [9], on the one side, and of the trigonometric Rosen-Morse potential [10], [11] on the other side, has been paid due attention. In effect, the observation of degeneracies in the spectrum of a given potential problem that appear patterned after a known regular Lie algebraic symmetry, as a rule awakes the expectation that the very same algebra may determine the symmetry of the interaction in question.

In the literature on exactly solvable potentials, degeneracy is usually interpreted as a consequence of some underlying potential algebra. Yet, it is well known a fact that at the same time quantum mechanics successfully describes also phenomena of degeneracy without symmetry, as is the violation of the non-crossing rule in the correlated electron system of Benzene described by means of the Hubbard Hamiltonian [12], or the detection of resonance degeneracies in a double well potential [13]. Such type of degeneracies are ordinarily termed to as accidental, better, fortuitous. An indispensable text on the aspects of degeneracy without symmetry is provided by [14] within the context of quantum chaotic motion. However, one should keep in mind that conditioning degeneracy by symmetry is not exclusive to regular Lie symmetries alone. Especially in non-linear systems, degeneracy can find explanation in terms of quantum group symmetries, obtained by deforming ordinary Lie algebras, as observed for degenerate Landau levels describing plane motion within a constant magnetic field of a charged massive spin-less particle [15]. Recapitulating the literature on exactly solvable potentials, degeneracies so far have been either associated with eigenvalue problems (in properly chosen variables) of differential Casimir operators of Lie symmetry algebras, regular or deformed, or, with the absence of such.

We here draw attention to a different option in showing that degeneracy conservation by perturbation can also be understood as equivalence between the matrix elements of an interaction Hamiltonian and the matrix elements of finite polynomials of a Casimir operator of a properly identified Lie algebra, the polynomial coefficients being interaction (potential) specific. Such is the case of the perturbation of the free quantum motion on  $S^3$  by the two-parameter potential,

$$V_{S^3}(\chi) = b^2 \sec^2 \chi - b(2\ell + 1) \tan \chi \sec \chi. \quad (1)$$

Here,  $\ell$  is discrete natural number, while the external scale introduced by the parameter  $b$  is continuous.

The paper is structured as follows. In the next section we briefly review for the sake of self-sufficiency of the presentation, the 1D Schrödinger equation with Scarf I with the emphasize on its hydrogen-like degeneracies for  $\ell$  non-negative integer, and review the proof of an  $so(4)$  potential algebra for the case of two quantized parameters. In section 3 we place the Scarf potential problem on  $S_R^{d+1}$  for any  $d$  although then we focus without loss of generality on  $S_R^3$  for the sake of concreteness. We show that the wave functions,  $\psi_{N\ell m}(\chi, \theta, \varphi)$ , of Scarf I (with  $N$  standing for the principal quantum number,  $N = (\ell + n + 1)$ , and  $n$  denoting the nodes of the wave function) do not transform irreducibly under  $so(4)$  because they turn to be mixtures of representation functions transforming

according to  $(K + 1)^2$ -dimensional carrier spaces of a properly identified  $so(4)$  algebra realization, with the value of the 4d angular momentum,  $K$ , varying as,  $K \in [\ell = (N - 1 - n), N - 1] \in \mathbf{N}$ . The above decompositions are simultaneously expansions in power series in the perturbation parameter,  $b$ , in which carrier spaces of dimensionality  $(K + 1)^2$  contribute up to the order  $\mathcal{O}(b^{N-1-K})$ . Also there we present the generalization of the notion of a Hamiltonian from a *single differential Casimir operator* (potential algebra concept [2]) to an equivalent diagonal matrix form whose elements are potential specific *finite polynomials* of the aforementioned Casimir operator (“dynamics governed by generalized symmetry algebra invariants”, abbreviated, “dynamical symmetry” [16]). We employ the latter concept in the explanation of the hydrogen-like degeneracy patterns in the spectrum of the Scarf I Hamiltonian. The closing section contains the summary of the results and discusses the perspectives.

## 2 The trigonometric Scarf potential and its degeneracy patterns

The (periodic) trigonometric Scarf potential is of frequent use in the description of di-atomic-, and poly-atomic molecules in solid-state physics, on the one side, or in di-molecular and poly-molecular systems in physical chemistry, on the other. In several quantum systems it simulates reasonably well the average effect exercised by the inter-atomic(inter-molecular) interactions on a single atom (molecule). The potential is characterized by two parameters only, is exactly solvable, and easy to handle with by computational soft-wares, all advantages that make it interesting to both theoretical studies and applications. Specifically in the present study, we focus on the peculiarity that the spectrum of Scarf I in (1) exclusively depends on the  $\ell$  parameter alone, while the importance of the  $b$  parameter confines to the level of the wave functions. For non-negative integer  $\ell$ -values, the latter spectrum shows typical hydrogen-like degeneracy and one expects the potential to have the geometric  $so(4)$  as a potential algebra. In such a case the Scarf I Hamiltonian would be linear in the Casimir operator of the  $so(4)$  algebra in a properly designed representation. This is true only restrictively, namely, only if the second parameter,  $b$ , which is irrelevant to the spectrum, has been properly quantized too. This case has been studied in the literature in great detail and the understanding has been gained that the corresponding wave functions transform as genuine  $so(4)$  representation functions [6]. However, for continuous  $b$  values this is not to be so, though the spectrum, in remaining unaffected, still keeps exhibiting those very same  $so(4)$  degeneracy patterns. Therefore, the case of Scarf I with one quantized and one continuous parameter may provide an intriguing template for studying the phenomenon of possibly observing Lie-algebraic degeneracies without an underlying geometric algebraic symmetry of the potential. To illuminate this issue, is the goal of the present study.

### 2.1 The general 1D Schrödinger equation with the two-parameter Scarf I potential

The 1D Schrödinger Hamiltonian,  $H_{\text{ScI}}(\chi)$ , with the trigonometric Scarf potential, here denoted by  $V_{\text{ScI}}(\chi)$ , and its exact solutions [6] are very well known and given (here in dimensionless units  $\hbar^2/2MR^2 = 1$ ) by

$$H_{\text{ScI}}(\chi) U(\chi) = \left[ -\frac{d^2}{d\chi^2} + V_{\text{ScI}}(\chi) \right] U(\chi) = \epsilon U(\chi), \quad (2)$$

$$V_{\text{ScI}}(\chi) = \frac{b^2 + a(a+1)}{\cos^2 \chi} - \frac{b(2a+1) \tan \chi}{\cos \chi}, \quad (3)$$

$$b^2 = \frac{2MR^2 B^2}{\hbar^2}, \quad a(a+1) = \frac{2MR^2}{\hbar^2} A(A-1), \quad (4)$$

$$U(\chi) = \mathbf{F}^{-1}(\chi) \cos^{a+1} \chi P_n^{a-b+\frac{1}{2}, a+b+\frac{1}{2}}(\sin \chi), \quad (5)$$

$$\mathbf{F}^{-1}(\chi) = \left( \frac{1 + \sin \chi}{1 - \sin \chi} \right)^{\frac{b}{2}} = e^{-b \tanh^{-1} \sin \chi}, \quad (6)$$

$$\epsilon = (a + n + 1)^2, \quad \epsilon = \frac{2MER^2}{\hbar^2}. \quad (7)$$

The Scarf I potential is determined by two-parameters, denoted by  $a$  and  $b$  when adimensional, or, by  $A$  and  $B$ , when carrying the dimensionality of energies. The dimensionless angular variable  $\chi$  is represented as,  $\chi = \frac{r}{R}$ , where  $r$  is a distance,  $R$  is a suited matching length parameter,  $E$  is the bound state energy in MeV,  $\epsilon$  stands for adimensional energy, and  $P_n^{\alpha, \beta}(\sin \chi)$  are the Jacobi polynomials.

## 2.2 The $so(4)$ algebra of Scarf I with two quantized parameters

The expression for the energy,  $\epsilon$ , in (7) is such that for a non-negative integer  $a = \ell \in \mathbb{N}$ , the spectrum exhibits a hydrogen-like  $N^2 = (\ell + n + 1)^2$ -fold degeneracy which is characteristic for an algebraic  $so(4)$  symmetry. This observation suggests  $H_{\text{ScI}}(\chi)$  to behave as a differential Casimir operator of a properly constructed  $so(4)$  algebra. To general proof of a symmetry of an interaction requires to

- identify the symmetry algebra of the emerging degeneracy patterns (recognized as  $so(4)$  in the case of interest),
- confirm irreducibility of the wave functions under transformations of same algebra.

For the potential under discussion this program has been executed in [6]. The line of reasoning is based on the fact that for discrete (quantized) parameters,  $a = m - \frac{1}{2}$ , and  $b = m'$ , the Jacobi polynomials in (5) defining the Scarf wave functions become,

$$P_n^{a-b+\frac{1}{2}, a+b+\frac{1}{2}}(\sin \chi) \longrightarrow P_n^{m-m', m+m'}(\sin \chi). \quad (8)$$

Then, the wave function  $U(\chi)$  in (5) allows for a factorization of Wigner's  $d_{mm'}^{j=n-m}$  functions, the representation functions of an  $su(2)$  algebra, according to,

$$\begin{aligned} U(\chi) &= \mathbf{G}^{-1}(\chi) d_{m'm}^{j=n-m}(\sin \chi), \\ \mathbf{G}^{-1}(\chi) &= \mathcal{N} \mathbf{F}^{-1}(\chi) \cos^{\frac{1}{2}} \chi \tan^{m'} \frac{\chi}{2}, \quad \chi \longrightarrow \frac{\pi}{2} - \chi, \end{aligned} \quad (9)$$

with  $\mathbf{F}^{-1}(\chi)$  from (6), and  $\mathcal{N}$  being a normalization constant. Such wave functions behave as representation functions of the following similarity transformed canonical rotational algebra,

$$\tilde{\mathbf{J}}^2(\chi, \varphi) = \mathbf{G}^{-1}(\chi) \mathbf{J}^2(\chi, \varphi) \mathbf{G}^{-1}(\chi), \quad (10)$$

with  $\chi$  now playing the rôle of the ordinary polar angle. This algebra has not been worked out explicitly in [6] but its ladder operators have been properly identified by inspection on the basis of the Schrödinger ladder operators,  $A^\pm = -\partial_\chi \pm W_{\text{ScI}}$ , with  $W_{\text{ScI}} = -(a-1) \tan \chi + b \sec \chi$  standing for the super-potential of  $V_{\text{ScI}}$ , and exploiting their property to factorize  $H_{\text{ScI}}$ . In a similar way, a second  $su(2)$  algebra has been constructed on the basis of the super-symmetric partner,  $\tilde{H}_{\text{ScI}}(\chi)$ . The discrete parameters,  $m$ , and  $m'$  have been introduced as auxiliary phases into the wave functions according to,  $U(\chi) \rightarrow \exp(im\alpha + im'\beta)U(\chi)$ . In effect, two sets of ladder operators have been designed, in turn labeled as left (L) and right (R) handed. Their algebras have then been closed to  $su(2)_L$ , and  $su(2)_R$  by  $(-i\partial_\alpha)$ , and  $(-i\partial_\beta)$ , respectively. As a result, left-handed  $su_L(2)$ , and right-handed  $su_R(2)$  algebras have been designed and their direct sum,  $su_L(2) \oplus su_R(2)$ , identified with the algebra of the universal cover  $SU_L(2) \otimes SU_R(2)$  of the group  $SO(4)$ . This algebra locally is isomorphic to  $so(4)$ , i.e.  $su_L(2) \oplus su_R(2) \simeq so(4)$ . Then the Scarf I Hamiltonian has been cast in the form of the corresponding Casimir operator, denoted by  $\mathcal{C}_2^{J_L J_R}$ , that expresses in terms of the respective squared left (L) and right (R) handed angular momentum operators,  $\mathbf{J}_L^2$ , and  $\mathbf{J}_R^2$  as

$$\mathcal{C}_2^{J_L J_R} = 2(\mathbf{J}_L^2 + \mathbf{J}_R^2) = 2\mathcal{K}^2, \quad (11)$$

with  $\mathcal{K}^2$  standing for the operator of the squared 4D angular momentum. The above scheme has been independently employed to establish the  $so(4)$  symmetry of the trigonometric Pöschl-Teller potential by Quesne in [9]. We here

closely followed precisely this very reference. Using the algebra locally isomorphic to  $so(4)$  from above has the advantage that one generates both the common and the projective representations. As a reminder, the group  $SO(4)$  is the quotient of the universal covering  $SU(2)_L \otimes SU(2)_R$  by the center  $Z$ , i.e.  $SO(4) \simeq SU_L(2) \otimes SU_R(2)/Z$ . It is interesting to notice that  $SO(4)$  can alternatively be viewed as the quotient of the Euclidean group  $E^+(4)$  by the group of translations, i.e.  $SO(4) \simeq E^+(4)/T$ , which would provide a different technique for the treatment the above problem [17].

To recapitulate, the requirement on algebraic  $so(4)$  symmetry of the Scarf I Hamiltonian imposes on its parameters stringent conditions of quantization. The property of the Schrödinger ladder operators to partake a closed Lie algebra under certain restrictions on the values of the potential parameters, provides any time that such is possible, a powerful method for the algebraic description of various quantum mechanical problems such as those related to the Coulomb-, the Harmonic-Oscillator and other interactions [7].

However, without the quantization of the  $b$  parameter, the equality in (8) is no longer valid and one is left with

$$U(\chi) = \mathbf{F}^{-1}(\chi) \cos^{\ell+1} \chi P_n^{\ell-b+\frac{1}{2}, \ell+b+\frac{1}{2}}(\sin \chi), \quad (12)$$

where from now onwards  $a$  will be quantized to non-negative integer as  $a = \ell$ . The Jacobi polynomials in the latter equation are such that neither Wigner functions, nor Gegenbauer polynomials can be in general factorized, a circumstance that strongly points towards serious difficulties in the construction of an explicit similarity transformation of the canonical geometric  $so(4)$  algebra towards the Scarf I Hamiltonian with one quantized and one continuous parameter.

In effect, the status of  $so(4)$  as a symmetry algebra of the Scarf I Hamiltonian is no longer obvious, although the spectrum, in being independent of  $b$ , remains unaltered. It is one of the goals of the present study to examine the relationship between the Scarf I Hamiltonian and

$$\tilde{\mathcal{K}}^2 = \mathbf{F}^{-1} \mathcal{K}^2 \mathbf{F}, \quad (13)$$

as suggested by the wave function in (12). For that purpose, placing the problem on a hyper-spherical surface turns to be helpful.

### 3 Perturbing the free quantum motion on $S^{d+1}$ by Scarf I with one quantized parameter

The free motion on the  $(d+1)$  dimensional unit hypersphere, to be denoted by  $S^{d+1}$  embedded within a  $(d+2)$  dimensional Euclidean space,  $E_{d+2}$ , is given by

$$\Delta_{S^{d+1}} Y_{K_{d+2} K_{d+1} \dots \ell m}(\chi, \eta, \dots, \theta, \varphi) = K_{d+2} (K_{d+2} + d) Y_{K_{d+2} K_{d+1} \dots \ell m}(\chi, \eta, \dots, \theta, \varphi), \quad (14)$$

where  $\Delta_{S^{d+1}}$  is the Laplace-Beltrami operator on the surface under consideration, defined as,

$$\Delta_{S^{d+1}} = -\frac{1}{\cos^d \chi} \frac{\partial}{\partial \chi} \cos^d \chi \frac{\partial}{\partial \chi} + \frac{\mathbf{K}_{d+1}^2}{\cos^2 \chi}. \quad (15)$$

Here,  $\mathbf{K}_{d+1}^2$  stands for the squared angular momentum operator in the Euclidean space,  $E_{d+1}$ , of one less dimension,  $\chi, \eta, \dots, \theta \in [-\frac{\pi}{2}, +\frac{\pi}{2}]$  are polar angles,  $\varphi \in [0, 2\pi]$  is the standard azimuthal angle,  $Y_{K_{d+2} K_{d+1} \dots \ell m}(\chi, \eta, \dots, \theta, \varphi)$  are the hyper-spherical harmonics on  $S^{d+1}$ , and  $K_{d+2-t}$  stand for the angular momentum values within an Euclidean spaces of  $t$  less dimensions, i.e in  $E_{d+2-t}$ . Finally,  $\ell, m$  are in turn the standard  $E_3$  and  $E_2$  angular momenta. Confining to the quasi-radial motion with wave function, denoted by  $R_{K_{d+2} K_{d+1}}(\chi)$ , and changing variable as,

$$R_{K_{d+2} K_{d+1}}(\chi) = \frac{U_{K_{d+2} K_{d+1}}(\chi)}{\cos^{\frac{d}{2}} \chi}, \quad (16)$$

amounts to the following one-dimensional Schrödinger equation,

$$\left[ -\frac{d^2}{d\chi^2} + \frac{(K_{d+1} + \frac{d-1}{2})^2 - \frac{1}{4}}{\cos^2 \chi} \right] U_{K_{d+2}K_{d+1}}(\chi) = \left[ K_{d+2}(K_{d+2} + d) + \frac{d^2}{4} \right] U_{K_{d+2}K_{d+1}}(\chi). \quad (17)$$

Comparison to (3) reveals (17) as the Scarf I potential problem for  $b = 0$ , and

$$a = K_{d+1} + \frac{d-1}{2} - \frac{1}{2}, \quad (18)$$

with  $a$  either integer, or semi-integer. In this manner, the explicit  $so(d+2)$  potential algebras of the  $\sec^2$  interaction of the one-dimensional Schrödinger equation have been made manifest. In now switching to the full Scarf I potential in (2)–(7) amounts to the following perturbed motion on  $S^{d+1}$ ,

$$\begin{aligned} \left[ -\frac{1}{\cos^d \chi} \frac{\partial}{\partial \chi} \cos^d \chi \frac{\partial}{\partial \chi} + \frac{b^2 + K_{d+1}^2}{\cos^2 \chi} - \frac{b(2K_{d+1} + d - 1) \tan \chi}{\cos \chi} \right] \phi_{K_{d+2}K_{d+1}}(\chi) \\ = \epsilon_{K_{d+2}} \phi_{K_{d+2}K_{d+1}}(\chi) \\ \epsilon_{K_{d+2}} = K_{d+2}(K_{d+2} + d) + \frac{d^2}{4}. \end{aligned} \quad (19)$$

Correspondingly, the solutions to (19) are read off from (5) as

$$\phi_{K_{d+2}K_{d+1}}(\chi) = \frac{U(\chi)}{\cos^{\frac{d}{2}} \chi}, \quad U(\chi) = \mathbf{F}^{-1}(\chi) \cos^{K_{d+1} + \frac{d-1}{2} + \frac{1}{2}} \chi P_n^{K_{d+1} + \frac{d-1}{2} - b, K_{d+1} + \frac{d-1}{2} + b}(\sin \chi), \quad (20)$$

with  $a$  from (18). Comparison to (14) shows that the perturbation retains the degeneracy patterns of the free motion in *any dimension* and rises the question on the symmetry of the full trigonometric Scarf potential.

In the following, we shall focus on  $S^3$ , setting  $d = 2$ , for concreteness, and without loss of generality. In so doing,  $K_4$  becomes the four-dimensional angular momentum value, to be denoted by  $K$  only, while  $K_{2+1}$  is no more but the ordinary angular momentum,  $\ell$ . With that, i.e. for  $a = \ell$ , the perturbation potential announced in (1) in the introduction becomes,

$$V_{S^3}(\chi) = \frac{b^2}{\cos^2 \chi} - \frac{b(2\ell + 1)}{\cos \chi} \tan \chi. \quad (21)$$

It is that very potential that will be referred to from now onwards as Scarf I on  $S^3$ . Apparently, for  $b = 0$  the free quantum motion on the curved surface under consideration is recovered. For  $d = 2$ , the equation (19) can also be viewed as the 4D quantum mechanical rigid rotator perturbed by  $V_{S^3}(\chi)$ , a problem of interest to di-atomic- or di-molecular systems. In furthermore recalling the relationship between the Laplace-Beltrami operator and the operator of the squared 4D angular momentum,  $\mathcal{K}^2$ , a Casimir invariant of the  $so(4)$  isometry algebra of  $S^3$ ,

$$-\Delta_{S^3} = \mathcal{K}^2, \quad (22)$$

allows to cast the free quantum motion on the unit hyper-sphere  $S^3$  in terms of the  $\mathcal{K}^2$  eigenvalue problem as

$$(\mathcal{K}^2 + 1) Y_{K\ell m}(\chi, \theta, \varphi) = (K + 1)^2 Y_{K\ell m}(\chi, \theta, \varphi). \quad (23)$$

Here,  $Y_{K\ell m}(\chi, \theta, \varphi)$  stand for the well-known 4D hyper-spherical harmonics, and with  $K \in [0, \infty)$ ,  $\ell \in [0, K]$ , and  $m \in [-\ell, +\ell]$ . The 4D hyper-spherical harmonics are the representation functions of the isometry  $so(4)$  algebra, which describe  $(K + 1)^2$ -dimensional  $so(4)$  carrier spaces and are defined according to,

$$Y_{K\ell m}(\chi, \theta, \varphi) = S_{K\ell}(\chi) Y_{\ell}^m(\theta, \varphi), \quad S_{K\ell}(\chi) = \cos^{\ell} \chi \mathcal{G}_{n=K-\ell}^{\ell+1}(\sin \chi). \quad (24)$$

Here,  $\mathcal{G}_{n=K-\ell}^{\ell+1}(\sin \chi)$  stand for the Gegenbauer polynomials. The  $S_{K\ell}(\chi)$  functions are sometimes referred to as the “quasi-radial” functions of the free motion [18].

Here,  $\mathcal{K}^2$  is expressed in terms of the six generators  $J_i$  and  $A_i$  with  $i = 1, 2, 3$ , spanning the  $so(4)$  algebra [19],

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad [A_i, A_j] = i\epsilon_{ijk}J_k, \quad [J_i, A_k] = i\epsilon_{ijk}A_k, \quad (25)$$

as

$$\mathcal{K}^2 = 2(\mathbf{J}^2 + \mathbf{A}^2). \quad (26)$$

In terms of (22), the equation of the perturbed motion on  $S^3$  which we will be dealing with here, takes the following final shape,

$$\begin{aligned} \mathcal{H}_{\text{Sc}}(\chi)\psi_{N\ell m}(\chi, \theta, \varphi) &= \epsilon_N \psi_{N\ell m}(\chi, \theta, \varphi), \\ \mathcal{H}_{\text{Sc}}(\chi) &= \mathcal{K}^2 + 1 + V_{S^3}(\chi), \\ \mathcal{K}^2 &= -\frac{1}{\cos^2 \chi} \frac{\partial}{\partial \chi} \cos^2 \chi \frac{\partial}{\partial \chi} + \frac{\mathbf{L}^2}{\cos^2 \chi}, \\ \psi_{N\ell m}(\chi, \theta, \varphi) &= \phi_{N\ell}(\chi) Y_\ell^m(\theta, \varphi) \\ &\equiv \mathbf{F}^{-1}(\chi) \cos^\ell \chi P_{n=N-1-\ell}^{\alpha, \beta}(\sin \chi) Y_\ell^m(\theta, \varphi), \\ N = n + \ell + 1, \quad \alpha = \ell - b + \frac{1}{2}, \quad \beta = \ell + b + \frac{1}{2}, \end{aligned} \quad (27)$$

with  $V_{S^3}$  from (21), and  $\mathbf{F}^{-1}(\chi)$  from (6). The energy excitations are,

$$\epsilon_N = N^2, \quad N \in \mathbf{N}, \quad N \in [1, \infty). \quad (28)$$

The remarkable aspect of the perturbation by the trigonometric Scarf potential is that despite the drastic change in the quasi-radial wave functions from unperturbed,  $S_{K\ell}(\chi)$  in (24), to perturbed,  $\phi_{N\ell}(\chi)$  in (27) according to,

$$S_{K\ell}(\chi) = \cos^\ell \chi \mathcal{G}_{n=K-\ell}^{\ell+1}(\sin \chi) \longrightarrow \phi_{N\ell}(\chi) = e^{-b \tanh^{-1} \sin \chi} \cos^\ell \chi P_{n=N-1-\ell}^{\alpha, \beta}(\sin \chi), \quad (29)$$

its spectrum remains independent of the external-scale introducing parameter  $b$ , which can be as well infinitesimally small, as finite. This spectrum is still characterized by that very same  $N^2$ -fold degeneracy of the states in a level, just as the hydrogen atom, and formally copies the  $(\mathcal{K}^2 + 1)$ -eigenvalue problem. From the equation (29) one immediately reads off that for the particular case of the parameter  $\ell$  taking its maximal value of  $\ell = (N - 1)$  (it includes the ground state,  $(N - 1) = \ell = 0$ ), the polynomials on both sides are of zero degree, i.e. constants, and one encounters equality between the Scarf I solutions on  $S^3$ , on the one side, and the  $\mathbf{F}^{-1}(\chi)$  transformed representation functions of the isometry  $so(4)$  algebra, on the other, namely,

$$\psi_{N, \ell=(N-1), m}(\chi, \theta, \varphi) = \mathbf{F}^{-1}(\chi) Y_{K=(N-1), \ell=(N-1), m}(\chi, \theta, \varphi) \equiv \tilde{Y}_{K=(N-1), \ell=(N-1), m}(\chi, \theta, \varphi), \quad (30)$$

holds valid. Notice that  $\tilde{Y}_{K=(N-1), \ell=(N-1), m}(\chi, \theta, \varphi)$  behave as representation functions of an  $so(4)$  algebra similarity transformed to,  $\mathbf{F}^{-1}\mathcal{K}^2\mathbf{F} = \tilde{\mathcal{K}}^2$ . For this particular case the solutions of the perturbed quantum motion on  $S^3$  under discussion result  $so(4)$  symmetric, though the algebra is in a representation that is unitarily nonequivalent to the hyper-spherical one. The equation (30) is suggestive of a relationship between the Scarf I Hamiltonian and a similarity transformation of the geometric  $so(4)$  algebra in (25) by the exponential function  $\mathbf{F}^{-1}(\chi)$  in (6), which also formed part of the design of the explicit potential algebra of Scarf I with two quantized parameters in (9). The next subsection is devoted to the construction of the transformed algebra (13) (with  $C_2^{J_L, J_R}$  being replaced by the squared 4D angular momentum operator,  $\mathcal{K}^2$ ) and to the comparison of its Casimir operator to the Scarf I Hamiltonian in (27).

### 3.1 Geometric $so(4)$ symmetry loss in the Scarf I potential problem on $S^3$ without lifting the degeneracy of the free quantum motion

We are interested in calculating the class of representation functions of the  $so(4)$  algebra to which  $\tilde{Y}_{K=(N-1), \ell=(N-1), m}(\chi, \theta, \varphi)$  in (30) belong. For this purpose we consider the algebra spanned by the set of elements,

$$\begin{aligned}\tilde{J}_i &= \mathbf{F}^{-1} J_i \mathbf{F}, & \tilde{A}_i &= \mathbf{F}^{-1} A_i \mathbf{F}, \quad i = 1, 2, 3, \\ \tilde{\mathcal{K}}^2 &= \mathbf{F}^{-1} \mathcal{K}^2 \mathbf{F}, & \tilde{\mathcal{K}}^2 &= 2 \sum_{i=1}^3 \left( \tilde{J}_i^2 + \tilde{A}_i^2 \right),\end{aligned}\tag{31}$$

with  $\mathbf{F}^{-1}(\chi)$  from (6). The corresponding representation functions will be termed to as exponentially rescaled hyper-spherical harmonics, defined as,

$$\begin{aligned}\tilde{Y}_{K\ell m}(\chi, \theta, \varphi) &= \mathbf{F}^{-1}(\chi) S_{K\ell}(\chi) Y_\ell^m(\theta, \varphi) = \tilde{S}_{K\ell}(\chi) Y_\ell^m(\theta, \varphi), \\ \tilde{S}_{K\ell}(\chi) &= \mathbf{F}^{-1}(\chi) S_{K\ell}(\chi).\end{aligned}\tag{32}$$

Next we calculate the similarity transformed Casimir operator,  $[\tilde{\mathcal{K}}^2 + 1] = [\mathbf{F}^{-1}(\chi) \mathcal{K}^2 \mathbf{F}(\chi) + 1]$ , together with its action on the  $\tilde{S}_{K\ell}(\chi)$ , functions i.e.

$$[\tilde{\mathcal{K}}^2 + 1] \tilde{S}_{K\ell}(\chi) = [\mathbf{F}^{-1}(\chi) \mathcal{K}^2 \mathbf{F}(\chi) + 1] \tilde{S}_{K\ell}(\chi).\tag{33}$$

In so doing we find the following general expression (see [2] for a similar procedure),

$$\begin{aligned}[\mathbf{F}^{-1}(\chi) \mathcal{K}^2 \mathbf{F}(\chi) + 1] \tilde{S}_{K\ell}(\chi) &= [\mathcal{K}^2 + 1 + \mathcal{V}(\chi)] \tilde{S}_{K\ell}(\chi), \\ \mathcal{V}(\chi) &= -\mathbf{F}^{-1}(\chi) \left[ \frac{\partial^2 \mathbf{F}(\chi)}{\partial \chi^2} - 2 \tan \chi \frac{\partial \mathbf{F}(\chi)}{\partial \chi} \right] \tilde{S}_{K\ell}(\chi) \\ &\quad - 2\mathbf{F}^{-1}(\chi) \frac{\partial \mathbf{F}(\chi)}{\partial \chi} \frac{\partial \tilde{S}_{K\ell}(\chi)}{\partial \chi}.\end{aligned}\tag{34}$$

The latter equation makes manifest that similarity transformations of algebra invariants generate potentials  $\mathcal{V}(\chi)$  which in general contain gradiental terms.

For the specific form of the function  $\mathbf{F}(\chi)$  defined in (6) one finds,

$$-\mathbf{F}^{-1}(\chi) \frac{\partial^2 \mathbf{F}(\chi)}{\partial \chi^2} = \left( -\frac{b^2}{\cos^2 \chi} + b \frac{\tan \chi}{\cos \chi} \right),\tag{35}$$

$$2 \tan \chi \mathbf{F}^{-1}(\chi) \frac{\partial \mathbf{F}(\chi)}{\partial \chi} = -2b \frac{\tan \chi}{\cos \chi},\tag{36}$$

$$\begin{aligned}-2\mathbf{F}^{-1}(\chi) \frac{\partial \tilde{S}_{K\ell}(\chi)}{\partial \chi} \frac{\partial \mathbf{F}(\chi)}{\partial \chi} &= \frac{2b^2}{\cos^2 \chi} \tilde{S}_{K\ell}(\chi) - \frac{2b\ell}{\cos \chi} \tan \chi \tilde{S}_{K\ell}(\chi) \\ &\quad + \frac{2b}{\cos \chi} \mathbf{F}^{-1}(\chi) \cos^\ell \chi \frac{\partial \mathcal{G}_n^{\ell+1}(\sin \chi)}{\partial \chi},\end{aligned}\tag{37}$$

where use has been made of

$$\frac{\partial \mathbf{F}(\chi)}{\partial \chi} = -\frac{b}{\cos \chi} \mathbf{F}(\chi).\tag{38}$$

Putting all together and back-substituting the eqs. (35)–(37) into (34) yields

$$\begin{aligned}(\tilde{\mathcal{K}}^2 + 1) \tilde{S}_{K\ell}(\chi) &= \left[ \mathcal{K}^2 + 1 + \frac{b^2}{\cos^2 \chi} - \frac{b(2\ell + 1)}{\cos \chi} \tan \chi \right] \tilde{S}_{K\ell}(\chi) \\ &\quad + \frac{2b}{\cos \chi} \mathbf{F}^{-1}(\chi) \cos^\ell \chi \frac{\partial \mathcal{G}_n^{\ell+1}(\sin \chi)}{\partial \chi},\end{aligned}\tag{39}$$



meaning that the similarity transformation of  $\mathcal{K}^2$  by  $\mathbf{F}^{-1}(\chi)$  gives only partially rise to the anticipated trigonometric Scarf potential,  $V_{S^3}(\chi)$  in (21), the rest being a gradiental term,

$$(\tilde{\mathcal{K}}^2 + 1) \tilde{S}_{K\ell}(\chi) = \mathcal{H}_{\text{Sc}}(\chi) \tilde{S}_{K\ell}(\chi) + \frac{2b}{\cos \chi} \mathbf{F}^{-1}(\chi) \cos^\ell \chi \frac{\partial \mathcal{G}_n^{\ell+1}(\sin \chi)}{\partial \chi}. \quad (40)$$

The latter equation equivalently rewrites to,

$$(K + 1)^2 \tilde{S}_{K\ell}(\chi) = \mathcal{H}_{\text{Sc}}(\chi) \tilde{S}_{K\ell}(\chi) + \frac{2b}{\cos \chi} \mathbf{F}^{-1}(\chi) \cos^\ell \chi \frac{\partial \mathcal{G}_n^{\ell+1}(\sin \chi)}{\partial \chi}. \quad (41)$$

The conclusion is, that  $\tilde{S}_{K\ell}(\chi)$  do not solve the  $\mathcal{H}_{\text{Sc}}(\chi)$ -eigenvalue problem, except for the  $\ell = (N - 1)$  case already discussed in the above equation (30). And vice versa, for the Scarf I solutions,  $\mathcal{H}_{\text{Sc}}(\chi) \phi_{N\ell}(\chi) = N^2 \phi_{N\ell}(\chi)$ , one finds,

$$N^2 \phi_{N\ell}(\chi) = (\tilde{\mathcal{K}}^2 + 1) \phi_{N\ell}(\chi) - \frac{2b}{\cos \chi} \mathbf{F}^{-1}(\chi) \cos^\ell \chi \frac{\partial P_n^{\alpha,\beta}(\sin \chi)}{\partial \chi}, \quad (42)$$

that they are no eigenfunctions to  $\tilde{\mathcal{K}}^2$  due to the non-commutativity of the Scarf I Hamiltonian and the Casimir operator of the transformed algebra,

$$[\mathcal{H}_{\text{Sc}}(\chi), \tilde{\mathcal{K}}^2] \neq 0. \quad (43)$$

Our case is that for a general  $\ell \neq (N - 1)$ , the wave functions  $\psi_{N\ell m}(\chi, \theta, \varphi)$  in (27) describing the perturbed motion on  $S^3$  behave as mixtures of the type  $\psi_{N\ell m}(\chi, \theta, \varphi) = \sum_{K=\ell}^{K=N-1} c_{K\ell}(b^\eta) \tilde{Y}_{K\ell m}(\chi, \theta, \varphi)$ , with  $\eta \in [0, n]$ . Such a property is bound to remain independent on the parametrization of the hypersphere by virtue of the model independence of the Lie algebras.

### 3.2 The case $\ell = (N - 2)$ as an illustrative example

We now take a closer look on  $\ell = (N - 2)$ , the state with the next-to highest orbital angular momentum value within the multiplet, in which case the wave function of interest is given by

$$\phi_{N(N-2)}(\chi) = \mathbf{F}^{-1}(\chi) \cos^{(N-2)} \chi P_1^{(N-2)-b+\frac{1}{2}, (N-2)+b+\frac{1}{2}}(\sin \chi). \quad (44)$$

The Jacobi polynomial allows for the following decomposition into Gegenbauer polynomials,

$$\begin{aligned} P_1^{(N-2)-b+\frac{1}{2}, (N-2)+b+\frac{1}{2}}(\sin \chi) &= -b + \frac{1}{2} (2N - 1) \sin \chi \\ &= -b \mathcal{G}_0^{N-1}(\sin \chi) + \frac{(2N - 1)}{4(N - 1)} \mathcal{G}_1^{N-1}(\sin \chi). \end{aligned} \quad (45)$$

In noticing that by the aid of eq. (24),

$$\cos^{N-2} \chi \mathcal{G}_0^{N-1}(\sin \chi) = S_{(N-2)(N-2)}(\chi), \quad \cos^{N-2} \chi \mathcal{G}_1^{N-1}(\sin \chi) = S_{(N-1)(N-2)}(\chi), \quad (46)$$

allows to equivalently rewrite eq. (44) as

$$\phi_{N(N-2)}(\chi) = -b \tilde{S}_{(N-2)(N-2)}(\chi) + \frac{(2N - 1)}{4(N - 1)} \tilde{S}_{(N-1)(N-2)}(\chi), \quad (47)$$

with  $\tilde{S}_{K\ell}(\chi) = \mathbf{F}^{-1}(\chi)S_{K\ell}(\chi)$  defined in (32), and standing for the quasi-radial representation functions of the transformed algebra in (31). In consequence, same relationship holds valid at the level of the total Scarf I wave-, and  $\tilde{\mathcal{K}}^2$  representation functions, the exponentially rescaled hyper-spherical harmonics in (32),

$$\psi_{N(N-2)m}(\chi, \theta, \varphi) = -b \tilde{Y}_{(N-2)(N-2)m}(\chi, \theta, \varphi) + \frac{(2N-1)}{4(N-1)} \tilde{Y}_{(N-1)(N-2)m}(\chi, \theta, \varphi). \quad (48)$$

In effect, we observe that the lower dimensional  $so(4)$  carrier space,  $\tilde{Y}_{(N-2)(N-2)m}(\chi, \theta, \varphi)$ , contributes to the order  $\mathcal{O}(b^1)$  to  $\psi_{N(N-2)m}(\chi, \theta, \varphi)$ , i.e.,

$$\psi_{N\ell m}(\chi, \theta, \varphi) = \sum_{K=\ell}^{K=N-1} c_{K\ell}(b^{N-1-K}) \tilde{Y}_{K\ell m}(\chi, \theta, \varphi), \quad \ell = N-2, \quad (49)$$

with the expansion coefficients  $c_{K\ell}(b^{N-1-K})$  being uniquely fixed through the decomposition of the Jacobi into Gegenbauer polynomials. Therefore, for  $\ell = (N-2)$ , the Scarf I Hamiltonian on  $S^3$  deviates to the order  $\mathcal{O}(b^1)$  from the geometric  $so(4)$  algebra Casimir in (31).

It is straightforward to verify that for any  $\ell \neq (N-1)$  the wave functions of a motion on  $S^3$ , perturbed by the trigonometric Scarf potential, always represent themselves as mixtures of  $so(4)$  representation functions corresponding to carrier spaces of different dimensionality. Examples are listed in the Table 1. The generalization of eq. (48) to any  $\ell$  reads

$$\psi_{N\ell m}(\chi, \theta, \varphi) = \sum_{K=\ell}^{K=N-1} c_{K\ell}(b^\eta) \tilde{Y}_{K\ell m}(\chi, \theta, \varphi), \quad \eta \in [0, n]. \quad (50)$$

Notice summation over the  $K$ -index defining the dimensionality of the  $so(4)$  carrier spaces. For the lowest  $\ell$  values,  $\ell = (N-1), (N-2), (N-3)$ , one finds  $\eta = (N-1-K)$ . Therefore, the wave functions constituting an  $N^2$  degenerate multiplet of Scarf I transform irreducibly solely under  $so(3)$ . Further symmetries are those relevant for any arbitrary degeneracy problem, like  $GL(N^2)$ , or,  $SO(N^2)$ , meaning that any linear combination of the states within the multiplet is an eigenstate to the Hamiltonian under investigation that belongs to same eigenvalue as the basis vectors.

Finally, a comment is in order on the reason for which the perturbation of the quantum motion on  $S^3$  by Scarf I nonetheless happens to conserve the  $so(4)$  degeneracies. Though  $\phi_{N\ell}(\chi)$  by themselves do not behave as  $(\tilde{\mathcal{K}}^2 + 1)$  eigenfunctions, the contributions of the gradient term are such that one ends up with the common  $N^2$  eigenvalue and  $N^2$ -fold degeneracies of the states in a level. Take for example the  $\ell = (N-2)$  case already considered in (47) above. Substitution into (41) amounts to:

$$\begin{aligned} (\mathcal{K}^2 + 1 + V_{S^3}(\chi)) \phi_{N(N-2)}(\chi) &= (\tilde{\mathcal{K}}^2 + 1) \phi_{N(N-2)}(\chi) - \frac{2b}{\cos \chi} \mathbf{F}^{-1}(\chi) \cos^\ell \chi \frac{\partial P_1^{\alpha, \beta}(\sin \chi)}{\partial \chi} \\ &= (N-1)^2 (-b) \tilde{S}_{(N-2)(N-2)}(\chi) \\ &\quad + N^2 \frac{(2N-1)}{4(N-1)} \tilde{S}_{(N-1)(N-2)}(\chi) \\ &\quad + (2N-1)(-b) \tilde{S}_{(N-2)(N-2)}(\chi) \\ &= N^2 \phi_{N(N-2)}(\chi). \end{aligned} \quad (51)$$

In the full space, obtained by multiplying (51) by  $Y_{N-2}^m(\theta, \varphi)$  from the right, the latter equation amounts to,

$$\begin{aligned} (\tilde{\mathcal{K}}^2 + 1) \psi_{N(N-2)m}(\chi, \theta, \varphi) &= \frac{2b}{\cos \chi} \mathbf{F}^{-1}(\chi) \cos^{N-2} \chi \frac{\partial P_1^{\alpha, \beta}(\sin \chi)}{\partial \chi} Y_{N-2}^m(\theta, \varphi) \\ &= N^2 \psi_{N(N-2)m}(\chi, \theta, \varphi). \end{aligned} \quad (52)$$

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$\phi_{N\ell}(\chi)$	$=$	$\sum_{K=\ell}^{K=N-1} c_{K\ell}(b^\eta) \tilde{S}_{K\ell}(\chi)$
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$\phi_{N(N-1)}(\chi)$	$=$	$\tilde{S}_{(N-1)(N-1)}(\chi)$
$\phi_{N(N-2)}(\chi)$	$=$	$\frac{2N-1}{4(N-1)} \tilde{S}_{(N-1)(N-2)}(\chi) - b \tilde{S}_{(N-2)(N-2)}(\chi)$
$\phi_{N(N-3)}(\chi)$	$=$	$\frac{1}{8} \frac{(2N-1)}{(N-2)} \tilde{S}_{(N-1)(N-3)}(\chi) - \frac{b}{2} \frac{(N-1)}{(N-2)} \tilde{S}_{(N-2)(N-3)}(\chi) + \frac{b^2}{2} \tilde{S}_{(N-3)(N-3)}(\chi)$
$\phi_{N(N-4)}(\chi)$	$=$	$\frac{1}{32} \frac{4(N-1)^2-1}{(N-3)(N-2)} \tilde{S}_{(N-1)(N-4)}(\chi) - \frac{b}{8} \frac{(2N-3)(N-1)}{(N-2)(N-3)} \tilde{S}_{(N-2)(N-4)}(\chi)$ $+ \frac{b^2}{8} \frac{(2N-3)}{(N-3)} \tilde{S}_{(N-3)(N-4)}(\chi)$ $- \frac{b}{24} \left[ \frac{4b^2(N-2)+(2N-1)}{(N-2)} \right] \tilde{S}_{(N-4)(N-4)}(\chi)$

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Table 1: Decompositions of “quasi-radial” wave functions,  $\phi_{N\ell}(\chi)$ , of Scarf I in eq. (27) in the basis of the “quasi-radial” parts,  $\tilde{S}_{K\ell}(\chi)$  of the exponentially rescaled hyper-spherical harmonics,  $\tilde{Y}_{K\ell m}(\chi, \theta, \varphi)$ , in (32). It is well visible that the Scarf I solutions are mixtures of representation functions describing  $so(4)$  carrier spaces of different dimensionality, thus making the geometric  $so(4)$  symmetry loss manifest. The decompositions are simultaneously finite power series in the symmetry breaking scale  $b$ . Notice that the leading order ( $\mathcal{O}(b^0)$ ) terms respect the symmetry of the unperturbed motion, as it should be. In this fashion, a quantitative scheme is elaborated which allows to keep track of the order to which the Scarf I Hamiltonian deviates from the  $so(4)$  Casimir in (31).

This simple exercise shows that the  $N^2$ -fold degeneracy in the spectrum of the trigonometric Scarf potential with the  $a$  parameter quantized to non-negative integer values can not be attributed in the usual way to a geometric  $so(4)$  algebraic symmetry. Yet, one is still left with the option of allowing for algebra invariants in the form of finite polynomials of the transformed Casimir operator  $\tilde{\mathcal{K}}^2$  in (31) and (40). If so, the rôle of the gradiental term in ending up with  $\phi_{N\ell}(\chi)$  functions belonging to same  $N^2$  eigenvalue, despite their evident decomposition into irreducible  $\tilde{\mathcal{K}}^2$  carrier space of different dimensionalities in (50), could be equally well played by properly chosen polynomial coefficients. Take again as an example the  $\ell = (N-2)$  case considered above. The polynomial invariant of the transformed  $so(4)$  algebra whose eigenvalue problem for  $\phi_{N(N-2)}(\chi)$  is identical to that of  $\mathcal{H}_{\text{Sc}}(\chi)$  on the same space, is given by

$$\left[ \left( \tilde{\mathcal{K}}^2 + 1 \right) \frac{\tilde{\mathcal{K}}^2 + 1 - (N-1)^2}{N^2 - (N-1)^2} - b \left( \tilde{\mathcal{K}}^2 + 1 + (2N-1) \right) \frac{\tilde{\mathcal{K}}^2 + 1 - N^2}{(N-1)^2 - N^2} \right] \left( \tilde{Y}_{(N-1)(N-2)m}(\chi, \theta, \varphi) + \right. \\ \left. \tilde{Y}_{(N-2)(N-2)m}(\chi, \theta, \varphi) \right) \equiv \mathcal{H}_{\text{Sc}} \psi_{N(N-2)m}(\chi, \theta, \varphi) = N^2 \psi_{N(N-2)m}(\chi, \theta, \varphi). \quad (53)$$

Indeed, the term proportional to the projector<sup>4</sup> on the  $\tilde{S}_{(N-2)(N-2)}(\chi)$  component in (51), i.e. on the  $(N-1)^2$  eigenvalue,  $\mathcal{P}^{[(N-1)^2]} = \frac{\tilde{\mathcal{K}}^2 + 1 - N^2}{(N-1)^2 - N^2}$ , provides *by construction* same contribution to  $(N-1)^2$ , namely,  $(2N-1)$ , as the gradiental term, thus allowing for the factorization of  $N^2$  as a net eigenvalue.

In a similar way, the eigenvalue of  $N^2$  to  $\phi_{N(N-3)}(\chi)$ , a space that contains according to the Table the two lower dimensional components,  $\tilde{S}_{(N-2)(N-3)}(\chi)$ , and  $\tilde{S}_{(N-3)(N-3)}(\chi)$ , can be understood as the following identity,

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<sup>4</sup>For the construction of such projectors see [20].

$$\begin{aligned}
& \left( (\tilde{\mathcal{K}}^2 + 1) \left[ \frac{\tilde{\mathcal{K}}^2 + 1 - (N-1)^2}{N^2 - (N-1)^2} \right] \left[ \frac{\tilde{\mathcal{K}}^2 + 1 - (N-2)^2}{N^2 - (N-2)^2} \right] \right. \\
& - \frac{b(N-1)}{2(N-2)} (\tilde{\mathcal{K}}^2 + 1 + (2N-1)) \left[ \frac{\tilde{\mathcal{K}}^2 + 1 - N^2}{(N-1)^2 - N^2} \right] \left[ \frac{\tilde{\mathcal{K}}^2 + 1 - (N-2)^2}{(N-1)^2 - (N-2)^2} \right] \\
& \left. + \frac{b^2}{2} (\tilde{\mathcal{K}}^2 + 1 + 4(N-1)) \left[ \frac{\tilde{\mathcal{K}}^2 + 1 - N^2}{(N-2)^2 - N^2} \right] \left[ \frac{\tilde{\mathcal{K}}^2 + 1 - (N-1)^2}{(N-2)^2 - (N-1)^2} \right] \right) \\
& \times \left( \tilde{Y}_{(N-1)(N-3)m}(\chi, \theta, \varphi) + \tilde{Y}_{(N-2)(N-3)m}(\chi, \theta, \varphi) + \tilde{Y}_{(N-3)(N-3)m}(\chi, \theta, \varphi) \right) \\
& \equiv \mathcal{H}_{\text{Sc}}(\chi) \psi_{N(N-3)m}(\chi, \theta, \varphi) = N^2 \psi_{N(N-3)m}(\chi, \theta, \varphi). \tag{54}
\end{aligned}$$

In this case the polynomial is of third order in the Casimir operator  $\tilde{\mathcal{K}}^2$  in (40). Along this line, appropriate Casimir polynomials for anyone of the degenerate states can be designed. The above perturbative construction amounts to a potential specific diagonal matrix representation of the Scarf I Hamiltonian on  $S^3$ .

Designing particle dynamics in terms of Hamiltonians as functions of certain Lie-algebra Casimir operators is of common use and great utility in nuclear physics, where the complexity of the systems presents an obstacle in the formulation of exactly soluble potential problems. This concept is of a fundamental importance within the powerful approach of the Interacting Boson Model (IBM) [16]. In the latter, such descriptions of degeneracies are termed to as “dynamics governed by generalized invariants of a symmetry algebra”, or, abbreviated, “dynamical symmetries” (not to be confused with the similar terminology used within the context of algebra realizations in the full phase space, as is the case of the  $H$ -Atom and the Runge-Lenz vector [21]). Conversely, the notion of a “potential algebra” confines to exactly solvable Hamiltonians which turn to be polynomials of first order in a standard differential Casimir operator. In this fashion, and in reference to the IBM terminology, the hydrogen-like degeneracy patterns in the spectrum under investigation can be understood as a promotion of the geometric  $so(4)$  potential algebra of the  $\sec^2$  interaction (corresponding to the free motion) to a dynamical  $so(4)$  symmetry of the Scarf I potential (corresponding to the perturbed motion), with the algebra being in a representation unitarily nonequivalent to the hyper-spherical one. Polynomials of the type in (53)-(54), when considered in terms of coefficients distributed at random around the Scarf I -specific values, would allow for the description of  $so(4)$  degeneracy patterns by means of wave functions fluctuating around the exact Scarf I solutions and could be of interest in simulation studies of systems with next-to Scarf I potential interactions. The scheme extends to any  $so(d+2)$ .

## 4 Conclusions and Perspectives

The present study has been devoted to the explanation of the emerging  $so(4)$ -type of degeneracy patterns in the spectrum of the di-atomic (di-molecular) trigonometric Scarf potential for the case in which the parameter  $a$  in (7) was allowed to take only non-negative integer values, while  $b$  remained unrestricted. We first demonstrated that though perturbations by  $V_{S^3}(\chi)$  in (21) of the free motion are degeneracy conserving, they lead to the loss of the geometric  $so(4)$  symmetry.

Our argumentation was based on the observation that the wave functions of the perturbed motion behaved as mixtures of functions, properly identified as genuine  $so(4)$  representation functions in (32), and which, as illustrated by (48), and the Table, transformed as finite linear combinations of  $so(4)$  carrier spaces of different dimensionality. Simultaneously, the decompositions presented themselves as finite power series expansions in the  $b$  parameter, which permitted to quantitatively keep track of the order to which the symmetry under discussion is gradually fading away. Though the wave functions considered constitute a  $N^2$ -fold degenerate multiplet, in not

behaving as eigenstates to the standard Casimir operator of the geometric  $so(4)$  under discussion, prevents the interpretation of this symmetry as the culprit for the observed degeneracy, unless the notion of an algebra invariant has not been generalized to an arbitrary function of the aforementioned Casimir operator. In such a case (53)-(54), we showed that the diagonal matrix element of the Scarf I Hamiltonian for a given  $\psi_{N\ell m}$  is indistinguishable from the  $\sum_{K=\ell}^{K=N-1} \tilde{Y}_{K\ell m} \rightarrow \psi_{N\ell m}$  transition matrix element of a polynomial of degree  $n = (N - 1 - \ell)$  in the Casimir operator of the similarity transformed hyper-spherical algebra in eq. (40). In this way, we presented an explicit example on promoting by external scales a manifest geometric  $so(d + 2)$  symmetry of an exactly solvable potential (the  $so(4)$  of  $\sec^2$ , in our case) to an effective algebraic symmetry of the dynamics following a perturbation that retains the degeneracy. Our findings are backed up by the established finite decompositions of the Jacobi polynomials, the key ingredients of the Scarf I solutions, in the basis of the Gegenbauer polynomials, the key ingredients of the canonical  $so(4)/so(d + 2)$  representation functions.

The present study points towards the possibility of non-standard  $so(4)/so(2 + d)$  realizations in the trigonometric Scarf potential problem, a further option being deformations of the algebras as defined on the full phase space, where the momentum space for the sphere could be elaborated along the lines in ref. [22]. Alternatively, such a study could also be worked out in the parametrization of the sphere via its stereographic projection on an ambient linear space of one more dimension, with the aim to find the complete set of integrals of motion, a scheme that has already been successfully elaborated for the related Kepler-Coulomb problem on hyper-spheres of any dimension [23] and which leads to algebra deformations.

To recapitulate, the goal of the present investigation has been to stress on the possibility of retaining degeneracy in the process of a perturbation. The general interest in such an observation lies in the possibility to withdraw in a process of perturbation a fundamental geometric Lie algebraic symmetries by scales, such as temperatures, masses, lengths, without leaving trace in the spectra. Such subtle symmetry losses remain undetectable at the level of the energy excitations but they have inevitably to show up in the disintegration modes of the system.

In view of the wide use of the hyper-spherical geometry in the description of many-body systems such as Brownian motion [24], coherent states [25] etc. we expect our findings reported here to acquire relevance.

## References

- [1] J. P. Elliott and P. G. Dawber, *Symmetry in Physics* (Oxford Univ. Press, London, 1985).
- [2] J. Wu and Y. Alhassid, *The potential group approach and hypergeometric differential equations*, J. Math. Phys. **31** (1990), 557-562;  
J. Wu, Y. Alhassid, and F. Gürsey, *Group theory approach to scattering: IV. Solvable potentials associated with  $so(2,2)$* , Ann. Phys. **196** (1989), 163-181.
- [3] Ugo Moschella, *Quantum fields on  $dS$  and  $AdS$* , Annales Henri Poincaré **4**, Suppl. 1 (2003), S319-S332.
- [4] C. Rasinariu, J. V. Mallow, and A. Gangopadhyaya, *Exactly solvable problems of quantum mechanics and their spectrum generating algebras*, C. Eur. J. Phys. **5** (2007), 111-134.
- [5] A. O. Barut, Akira Inomata, and Raj Wilson, *A new realization of dynamical groups and factorization method*, J. Phys. A:Math.Gen. **20** (1987), 4075-4082.
- [6] G. Lévai, F. Cannata, and A. Ventura,  *$PT$ -symmetric potentials and the  $so(2,2)$  algebra*, J. Phys. A:Math.Gen. **35** (2002), 5041-5057.
- [7] D. Martinez, J. C. Flores-Urbina, R. D. Mota, and V. D. Granada, *The  $su(1,1)$  dynamical algebra from the Schrödinger ladder operators for  $N$ -dimensional systems: hydrogen atom, Mie-type potential, harmonic oscillator and pseudo-harmonic oscillator*, J. Phys. A:Math.Theor. **43** (2010), 135201.
- [8] N. Leija-Martinez, D. E. Alvarez-Castillo, and M. Kirchbach, *Breaking pseudo-rotational symmetry through  $H_+^2$  metric deformation in the Eckart potential problem*, Symm.Int.Geom.:Meth.Appl.(SIGMA) **7** (2011), 113.

- [9] C. Quesne, *An  $SL(4, R)$  Lie algebraic treatment of the first family of Pöschl-Teller potentials*, J. Phys. A:Math.Gen. **21** (1988), 4487–4500.
- [10] A. Pallares-Rivera and M. Kirchbach, *Symmetry and degeneracy of the curved Coulomb potential on the  $S^3$  ball*, J. Phys. A:Math.Theor. **44** (2011), 445302.
- [11] D. E. Alvarez-Castillo, C. B. Compean, and M. Kirchbach, *Rotational symmetry and degeneracy: a cotangent perturbed rigid rotator of unperturbed level multiplicity*, Mol. Phys. **109** (2011), 1477-1483.
- [12] Ole J. Heilmann and Elliot H. Lieb, *Violation of the non-crossing rule: The Hubbard Hamiltonian for Benzene*, Annales of the New York's Academy of Sciences **172** (1971), 584-617.
- [13] E. Hernandez. A. Jáuregui, and A. Mondragon, *Degeneracy of resonances in a double barrier potential*, J. Phys. A:Math.Gen. **33** (2000), 4507-4523 .
- [14] M. Berry, *Aspects of Degeneracy*, in "Chaotic behavior in quantum systems", ed. Giulio Casati, 123-140 (Plenum Press, New York, 1985).
- [15] A. Jellal, M. Daoud, and Y. Hassouni, *Supersymmetric sine algebra and degeneracy of Landau levels*, Phys. Lett. B **474** (2000) 122-129.
- [16] F. Iachello and P. Van Isacker, *The Interacting Boson Model* (Cambridge Univ. Press, Cambridge, 2005).
- [17] Willard Miller Jr, *Some applications of representation theory of the Euclidean group in three space*, Communications of Pure and Applied Mathematics, vol. XVII, 527-540 (1964).  
B. R. MacGregor, A. E. McCoy and S. Wickramasekara, *Unitary representations of the Galilean line group: Quantum mechanical principle of equivalence*, E-print arXiv:1107.2442V1[math-phys].
- [18] E. Kalnins, W. Miller, and G. S. Pogosyan, *The Coulomb-oscillator relation on  $n$ -dimensional spheres*, Physics of Atomic Nuclei **65** (2002), 1086-1094.
- [19] M. J. Englefield, *Group Theory and the Coulomb Problem* (Wiley-Interscience, a Division of John Wiley & Sons, Inc., N.Y, 1972).
- [20] Wu-Ki Tung, *Group Theory in Physics* (World Scientific, Singapore, 1985).
- [21] P. W. Higgs, *Dynamical symmetries in a spherical geometry*, J. Phys A:Math.Gen. **12**, 309-323 (1979).
- [22] M. A. Alonso, G. S. Pogosyan, and K. B. Wolf, *Wigner functions for curved spaces.II. On spheres*, J. Math. Phys. **44**, 1472-1489 (2003).
- [23] A. Ballesteros and F. J. Herranz, *Maximal superintegrability of the generalized Kepler-Coulomb system on  $N$ -dimensional curved spaces*, J. Phys. A:Math. Theor. **42**, 245203 (2009).
- [24] Jarl Nissfolk, Tobias Ekholm, and Chritse Elvinson, *Brownian dynamics simulations on a hypersphere in 4-space*, J. Chem. Phys. **119** (2003), 6423–6432;  
Jean-Michel Caillol, *Random Walks on Hyperspheres of Arbitrary Dimension*, E-Print arXiv: cond-mat/0401209
- [25] S. Cruz y Cruz, S. Kuru, and J. Negro, *Classical motion and coherent states for Pöschl-Teller potentials*, Phys. Lett. A **372** (2008), 1391-1405.